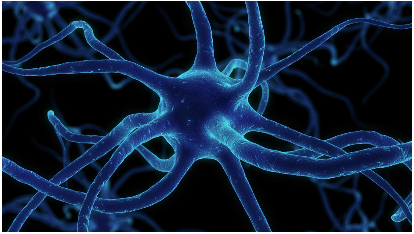
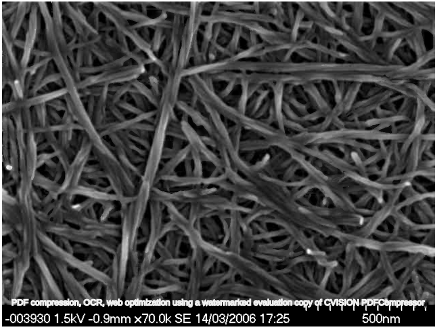
Drawing a graph in 3 pages

within its isotopy class in linear time

Vitaliy Kurlin, http://kurlin.org
Microsoft Research Cambridge
and Durham University, UK

Models of knotted structures

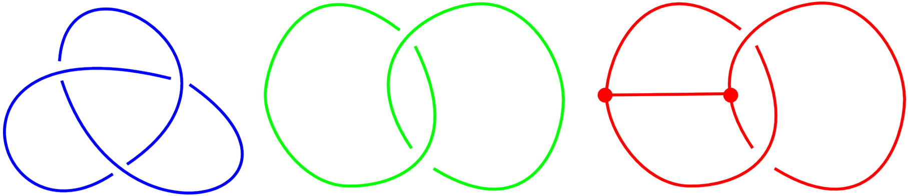




Knotted structured in nature are graphs with vertices, not simple closed loops like knots/links.

Spatial (knotted) graphs in $\mathbb{R}^3$

Def: a spatial graph is an embedding of a finite non-oriented graph $f : G \to \mathbb{R}^3$. So the image $f(G)$ has no self-intersections, but may have double crossings under a planar projection.

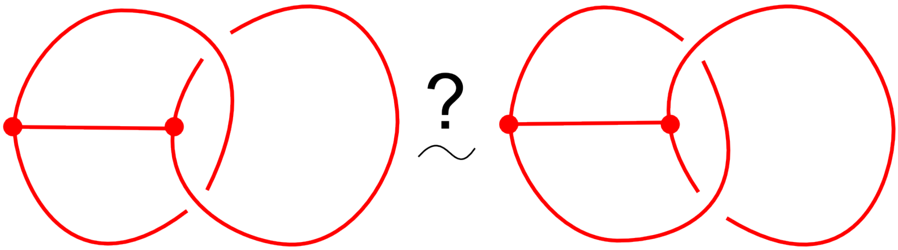


If $G \approx S^1$, the spatial graph $S^1 \subset \mathbb{R}^3$ is a knot.

If $G \approx \sqcup_{i=1}^m S_i^1$, the spatial graph is called a link.

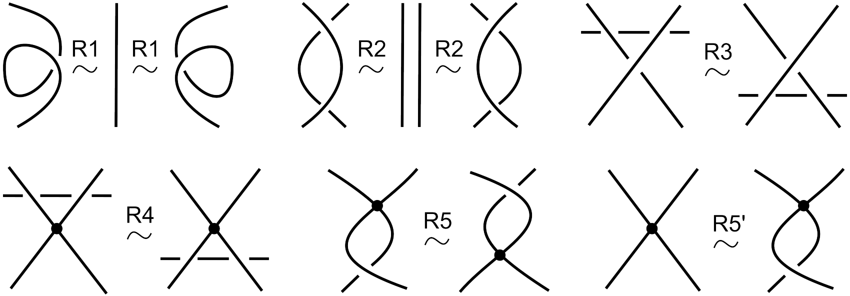
Isotopy of spatial graphs

Def: an ambient isotopy between spatial graphs $G, H \subset \mathbb{R}^3$ is a continuous family of ambient homeomorphisms $F_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0, 1]$, where $F_0 = \mathbf{id}$ on $\mathbb{R}^3$ and $F_1(G) = H$.



Reidemeister moves

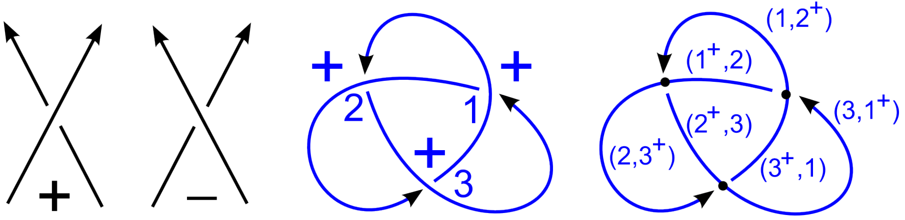
Thm: diagrams represent isotopic graphs *if and only if* they are connected by a plane isotopy and finitely many moves of these 5 types:



Move *R*5 is only for rigid graphs that have a neighborhood of any vertex in a moving plane.

Gauss code of a knot diagram

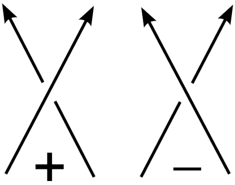
Def : fix orientations of edges, label crossings by $1, \dots, n$. Each crossing has a sign $\varepsilon \in \{\pm\}$.

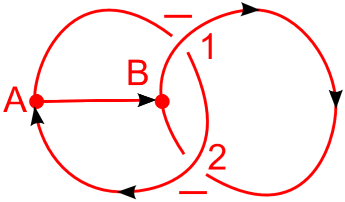


To get a Gauss code, starting from a base point write a word for each edge: i for i-th overpass, i^ε for i-th underpass. Trefoil: $12^+31^+23^+$.

Gauss code of a graph diagram

For a diagram of $G \subset \mathbb{R}^3$ we label vertices and encode crossing along each edge of G, note a cyclic order of edges at a vertex of deg > 3.





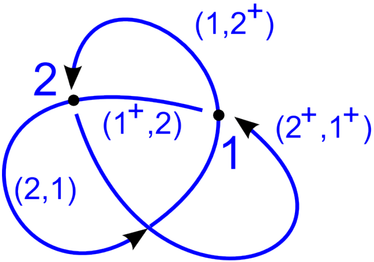
The red graph has 3 words AB, $A1^{-}2A$, $B12^{-}B$.
A Gauss code determines a plane diagram and is not unique in the isotopy class of $G \subset \mathbb{R}^3$.

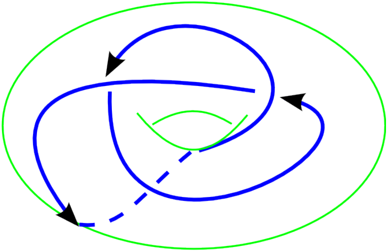
Realizability of Gauss codes

An abstract Gauss may not be realizable in $\mathbb{R}^2$.

Code $12^{+}1^{+}2$ has a diagram on a torus, not $\mathbb{R}^2$.

One approach is to allow any *virtual* crossings.





To draw a graph starting from a code, we solve **planarity problem**: which codes are realizable?

Graph $G(W)$ of a Gauss code W

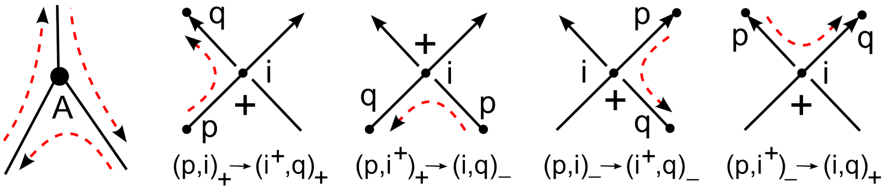
For an abstract Gauss code W with m letters $A, B, C, \ldots$ (for vertices) and 2n symbols from $\{i, i^+, i^-\}$, $i = 1, \ldots, n$ (for crossings), we build the graph $G(W)$ with $m + n$ vertices labeled by $A, B, C, \ldots$ and $1, 2, \ldots, n$ (without signs).

Connect vertices p, q by an edge in $G(W)$ if p, q (possibly with signs) are adjacent in W.

$W = 12^+31^+23^+$, $G(W)$ is a 'doubled' triangle with 2 edges between pairs of vertices $1, 2, 3$.

Carter surface of a Gauss code

Def : define cycles in $G(W)$ by the rule 'always turn left'. The Carter surface $S(W)$ is obtained by attaching a disk along each cycle in $G(W)$.

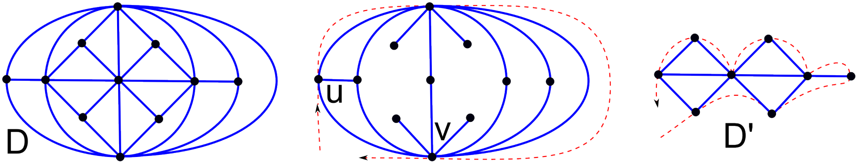


A Gauss code W is realizable $\Leftrightarrow$ $S(W) = S^2$.

Linear time algorithm: check that $\chi(S(W)) = 2$.

2-page embedding in linear time

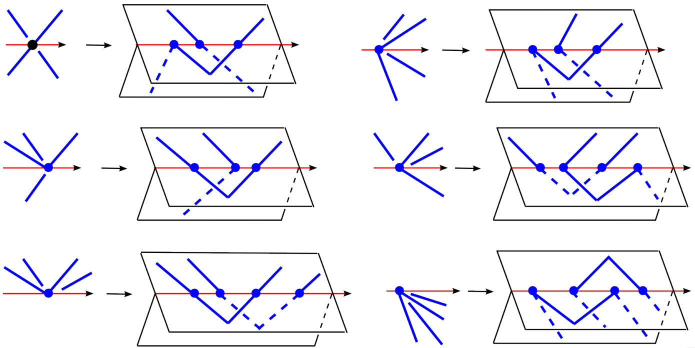
Th (G.L.M.S. ISAAC'07): any planar graph G can be embedded into $\mathbb{R}^2$ with all $|V|$ vertices in x-axis and max 1 bend per edge in time $O(|V|)$.



This max non-hamiltonian graph requires bends.

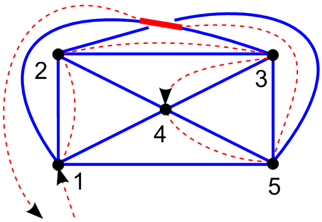
3-page embedding in linear time

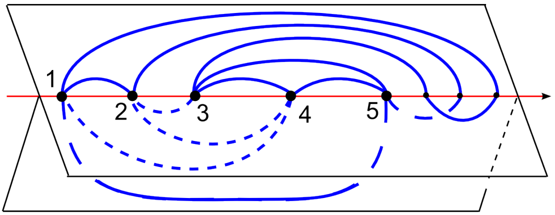
Th (VK'14): for a Gauss code W of a graph $G \subset \mathbb{R}^3$, an algorithm of complexity $O(|W|)$ draws a 3-page embedding of a graph H isotopic to G having max $8|W|$ intersections.



A 3-page embedding $K_5 \subset T_3 \times \mathbb{R}$

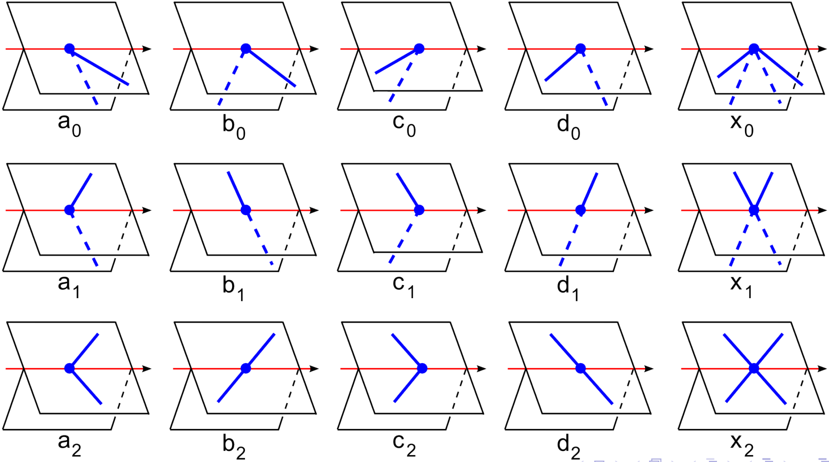
Theoretic construction: draw a curve α without any self-intersections that passes through each vertex and crossing of a diagram, deform $\mathbb{R}^2$ to make α straight, push overpasses into page 3.





Why 3-page embeddings?

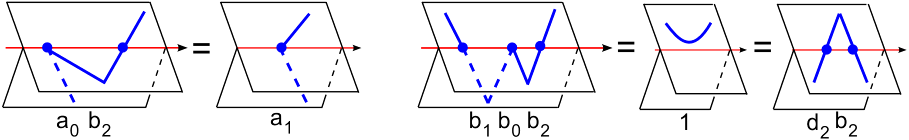
3-page embeddings are encoded by words in this alphabet (similarly for vertices of deg $\neq 4$)



Isotopy of graphs $\Leftrightarrow$ word problem

Th (VK'07): all central elements of a finitely presented semigroup *uniquely encode* all isotopy classes of graphs with vertices deg $\leq n$.

Typical local relations on 3-page embeddings:



15 generators, 84 relations for singular knots.

Summary and future work

- all spatial graphs in $\mathbb{R}^3$ are encoded by Gauss codes with a linear decision time
- any spatial graph $G \subset \mathbb{R}^3$ is isotopic to a 3-page embedding found in linear time

- from real data to theoretical models: recognize knots from a noisy 3D cloud, or from a 2D image (more challenging)
- collaboration is welcome! `kurlin.org/blog`